Nonlocal symmetries and conservation laws of the coupled Hirota equation

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Abstract

Using the lax pair, nonlocal symmetry of the coupled Hirota equation is obtained. By introducing an appropriate auxiliary dependent variable the nonlocal symmetry is successfully localized to a Lie point symmetry. For the closed prolongation, one-dimensional optimal systems and nonlocal conservation laws of the coupled Hirota equation are studied.

Keywords: Nonlocal symmetry, Optimal system, Conservation laws.

1. Introduction

The Lie symmetries[1–6] and their various generalizations have become an important subject in mathematics and physics. One can reduce the dimensions of partial differential equations(PDEs) and proceed to construct analytical solutions by using classical or non-classical Lie symmetries. However, with all its importance and power, the traditional Lie approach does not provide all the answers to mounting challenges of the modern nonlinear physics. In the 80s of the last century, there exist so-called nonlocal symmetries which entered the literature largely through the work of Olver[7]. Compared with the local symmetries, little importance is attached to the existence and applications of the nonlocal ones. The reason lies in that nonlocal symmetries are difficult to find and similarity reductions cannot be directly calculated. Many researchers[8–10] have done a lot of work in this area, references[11–13] give a direct way to solve this problem which so-called localization method of nonlocal symmetries. I.e. the original system is prolonged to a larger system such that the nonlocal symmetry of the original model becomes a local one of the prolonged system. When we get the Lie symmetries of prolonged system, there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is not usually feasible to list all possible group-invariant solutions to the system. Hence, the optimal system[2, 14–17] of group-invariant solutions should be constructed.

Conservation laws are used for the development of appropriate numerical methods and for mathematical analysis, in particular, existence,uniqueness and stability analysis. It can lead to some new integrable systems via reciprocal transformation. The famous Noether's theorem[18] provides a systematic way of determining conservation laws, for Euler-Lagrange differential equations, to each Noether symmetry associated with the Lagrangian there corresponds a conservation law which can be determined explicitly by a formula. But this theorem relies on the availability of classical Lagrangians. To find conservation laws of differential equations without classical

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Lagrangians, researchers have made various generalizations of Noether's theorem. Steudel[19] writes a conservation law in characteristic form, where the characteristics are the multipliers of the differential equations. In order to determine a conservation law one has to also find the related characteristics. Anco and Bluman[20] provides formulae for finding conservation laws for known characteristics. Infinitely many nonlocal conservation laws for(1+1)-dimensional evolution equations are revealed by Lou[21]. Symmetry considerations for PDEs were incorporated by Ibragimov[22] which can be computed by a formula.

This paper is arranged as follows: In Sec.2, the nonlocal symmetries of the coupled Hirota equation are obtained by using the Lax pair. In Sec.3, we transform the nonlocal symmetries into Lie point symmetries. Then, the finite symmetry transformations are obtained by solving the initial value problem. In Sec.4, an optimal system is constructed to classify the group-invariant solutions of the coupled Hirota equation. In Sec.5, based on the symmetries of prolonged system, nonlocal conservation laws of the coupled Hirota equation are given out. Finally, some conclusions and discussions are given in Sec.6.

2. Nonlocal symmetries of the coupled Hirota equation

The well-known the Hirota equation[23, 24] reads

$$iu_t + \alpha(u_{xx} + 2\,|u|^2\,u) + i\beta(u_{xxx} + 6\,|u|^2\,u_x) = 0, \tag{1}$$

α,β are real constants. Eq.(1) is the third flow of the nonlinear Schrödinger (NLS)hierarchy which can be used to describe many kinds of nonlinear phenomenas or mechanisms in the fields of physics,optical fibers,electric communication and other engineering sciences. Eq.(1)reduces to NLS equation when $\alpha = 1, \beta = 0$.

In this section, we shall consider the coupled Hirota equations

$$\begin{aligned} &iu_t + \alpha(u_{xx} - 2u^2v) + i\beta(u_{xxx} - 6uvu_x) = 0,\\ &iv_t - \alpha(v_{xx} - 2v^2u) + i\beta(v_{xxx} - 6uvv_x) = 0, \end{aligned} \tag{2}$$

Eqs.(2)are reduced to the Eq.(1)when $u = -v^*$, and $*$ denotes the complex conjugate.

The Lax pair of Eq.(2) has been obtained in[25]

$$\Phi_x = U\Phi, U = \begin{pmatrix} -i\lambda & u \\ v & i\lambda \end{pmatrix} \tag{3}$$

and

$$\Phi_t = V\Phi, V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \tag{4}$$

with

$$\begin{aligned} a &= -4\beta i\lambda^3 - 2\alpha i\lambda^2 - 2\beta iuv\lambda - \alpha iuv + \beta(vu_x - uv_x),\\ b &= 4\beta u\lambda^2 + (2\beta iu_x + 2\alpha u)\lambda + \alpha iu_x - \beta(u_{xx} - 2u^2v),\\ c &= 4\beta v\lambda^2 - (2\beta iv_x - 2\alpha v)\lambda - \alpha iv_x - \beta(v_{xx} - 2v^2u). \end{aligned} \tag{5}$$

where u and v are two potentials, the spectral parameter λ is an arbitrary complex constant and its eigenfunction is $\Phi = (\phi, \psi)^T$.

To seek for the nonlocal symmetries, we adopt a direct method[26]. First of all, the symmetries σ_1, σ_2 of the coupled Hirota equations are defined as solutions of their linearized equations

$$\begin{aligned}
&\sigma_{1t} - \alpha\sigma_{1xx}i + 4i\alpha\sigma_1 uv + 2i\alpha u^2\sigma_2 + \beta\sigma_{1xxx} - 6\beta\sigma_1 vu_x - 6\beta u\sigma_2 u_x - 6\beta uv\sigma_{1x} = 0,\\
&\sigma_{2t} + \alpha\sigma_{2xx}i - 4i\alpha uv\sigma_2 - 2i\alpha v^2\sigma_1 + \beta\sigma_{2xxx} - 6\beta\sigma_2 uv_x - 6\beta uv\sigma_{2x} - 6\beta v\sigma_1 v_x = 0,
\end{aligned} \tag{6}$$

which means equation (2) is form invariant under the infinitesimal transformations

$$\begin{aligned}
&u \to u + \epsilon\sigma_1,\\
&v \to v + \epsilon\sigma_2,
\end{aligned} \tag{7}$$

with the infinitesimal parameter ϵ.

The symmetry can be written as

$$\begin{aligned}
\sigma_1 &= X(x,t,u,v,\phi,\psi)u_x + T(x,t,u,v,\phi,\psi)u_t - U(x,t,u,v,\phi,\psi),\\
\sigma_2 &= X(x,t,u,v,\phi,\psi)v_x + T(x,t,u,v,\phi,\psi)v_t - V(x,t,u,v,\phi,\psi),
\end{aligned} \tag{8}$$

Here, X, T, U, V dependent on the auxiliary variables ϕ and ψ, so one may obtain some different results from Lie symmetries. Substituting Eq.(8) into Eq.(6) and eliminating $u_t, v_t, \phi_x, \phi_t, \psi_x, \psi_t$ in terms of the closed system, it yields a system of determining equations for the functions X, T, U, V, which can be solved by virtue of Maple to give

$$\begin{aligned}
&X = \tfrac{c_1 x}{3} + \tfrac{2\alpha^2 c_1 t}{9\beta} + c_3,\\
&T(x,t,u,v,\phi,\psi) = c_1 t + c_2,\\
&U(x,t,u,v,\phi,\psi) = \tfrac{i\alpha c_1 ux}{9\beta} + c_5 u + c_4\phi^2,\\
&V(x,t,u,v,\phi,\psi) = \tfrac{((-6c_1 - 9c_5)v + 9c_4\psi^2)\beta - i\alpha v c_1 x}{9\beta}
\end{aligned} \tag{9}$$

where $c_i (i = 1, ..., 5)$ are five arbitrary constants and $i^2 = -1$. It can be seen from the results(9), the results contain a nonlocal symmetry when $c_4 \neq 0$.

3. Localization of the nonlocal symmetry

As we all know, the nonlocal symmetries cannot be used to construct explicit solutions directly. Hence, one need to transform the nonlocal symmetries into local ones[11–13]. In this section, a related system which possesses a Lie point symmetry that is equivalent to the nonlocal symmetry will be found.

For simplicity, we let $c_1 = c_2 = c_3 = c_5 = 0, c_4 = -1$ in formula (9), i.e.,

$$\begin{aligned}
&\sigma_1 = \phi^2,\\
&\sigma_2 = \psi^2.
\end{aligned} \tag{10}$$

To localize the nonlocal symmetry (10), we have to solve the following linearized equations

$$\begin{aligned}&\sigma_{3x}+i\lambda\sigma_3-\sigma_1\phi-u\sigma_4=0,\\&\sigma_{4x}-\sigma_2\varphi-v\sigma_3-i\lambda\sigma_4=0,\end{aligned}\tag{11}$$

which means that Eqs.(3) invariant under the infinitesimal transformations

$$\begin{aligned}&\phi\rightarrow\phi+\epsilon\sigma_3,\\&\psi\rightarrow\psi+\epsilon\sigma_4,\end{aligned}\tag{12}$$

with σ_1,σ_2 given by (10). It is not difficult to verify that the solutions of (11) have the following forms

$$\sigma_3=\phi f,\quad \sigma_4=\psi f,\tag{13}$$

where f is given by

$$\begin{aligned}&f_x=\phi\psi,\\&f_t=-\beta\phi^2v_x+12\beta\lambda^2\phi\psi+4\lambda\alpha\phi\psi-\beta\psi^2u_x+2\beta\phi\psi uv+4i\lambda\beta\psi^2u-4i\lambda\beta\phi^2v+i\alpha\psi^2u-i\alpha\phi^2v.\end{aligned}\tag{14}$$

It is easy to obtain the following result

$$\sigma_5=\sigma_f=f^2.\tag{15}$$

The results (13) and (15) show that the nonlocal symmetry (10) in the original space $\{x,t,u,v\}$ has been successfully localized to a Lie point symmetry in the enlarged space $\{x,t,u,v,\phi,\psi,f\}$ with the vector form

$$V_1=\phi^2\frac{\partial}{\partial u}+\psi^2\frac{\partial}{\partial v}+\phi f\frac{\partial}{\partial\phi}+\psi f\frac{\partial}{\partial\psi}+f^2\frac{\partial}{\partial f}.\tag{16}$$

After succeeding in making the nonlocal symmetry(10) equivalent to Lie point symmetry (16) of the related prolonged system, we can construct the explicit solutions naturally by Lie group theory. With the Lie point symmetry(16), by solving the following initial value problem

$$\begin{aligned}&\tfrac{d\overline{u}}{d\epsilon}=\phi^2,\quad \overline{u}|_{\epsilon=0}=u,\\&\tfrac{d\overline{v}}{d\epsilon}=\psi^2,\quad \overline{v}|_{\epsilon=0}=v,\\&\tfrac{d\overline{\phi}}{d\epsilon}=\phi f,\quad \overline{\phi}|_{\epsilon=0}=\phi,\\&\tfrac{d\overline{\psi}}{d\epsilon}=\psi f,\quad \overline{\psi}|_{\epsilon=0}=\psi,\\&\tfrac{d\overline{f}}{d\epsilon}=f^2,\quad \overline{f}|_{\epsilon=0}=f,\end{aligned}\tag{17}$$

the finite symmetry transformation can be calculated as

$$\overline{u}=\frac{\epsilon fu-\varepsilon\phi^2-u}{\epsilon f-1},\overline{v}=\frac{\epsilon fv-\epsilon\psi^2-v}{\epsilon f-1},\overline{\phi}=\frac{\phi}{\varepsilon f-1},\overline{\psi}=\frac{\psi}{\epsilon f-1},\overline{f}=\frac{f}{\epsilon f-1},\tag{18}$$

For a given solution u,v,ϕ,ψ,f of Eqs.(18), above finite symmetry transformation will arrive at another so-

lution $\overline{u}, \overline{v}$. To search for more similarity reductions of Eqs.(2), one should consider Lie point symmetries of the whole prolonged system and assume the vector of the symmetries has the form

$$V = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} + P\frac{\partial}{\partial p} + Q\frac{\partial}{\partial q} + F\frac{\partial}{\partial f}, \tag{19}$$

which means that the closed system is invariant under the infinitesimal transformations

$$(x, t, u, p, q, f) \to (x + \epsilon X, t + \epsilon T, u + \epsilon U, \phi + \epsilon P, \psi + \epsilon Q, f + \epsilon F),$$

with

$$\begin{aligned}
\sigma_1 &= X(x,t,u,v,\phi,\psi,f)u_x + T(x,t,u,v,\phi,\psi,f)u_t - U(x,t,u,v,\phi,\psi,f),\\
\sigma_2 &= X(x,t,u,v,\phi,\psi,f)v_x + T(x,t,u,v,\phi,\psi,f)v_t - V(x,t,u,v,\phi,\psi,f),\\
\sigma_3 &= X(x,t,u,v,\phi,\psi,f)\phi_x + T(x,t,u,v,\phi,\psi,f)\phi_t - P(x,t,u,v,\phi,\psi,f),\\
\sigma_4 &= X(x,t,u,v,\phi,\psi,f)\psi_x + T(x,t,u,v,\phi,\psi,f)\psi_t - Q(x,t,u,v,\phi,\psi,f),\\
\sigma_5 &= X(x,t,u,v,\phi,\psi,f)f_x + T(x,t,u,v,\phi,\psi,f)f_t - F(x,t,u,v,\phi,\psi,f).
\end{aligned} \tag{20}$$

with $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ satisfy the linearized equations of Eqs.(2,3,4,14). Omitted here because of the formulas are long-winded.

Substituting Eq.(20) into linearized equations and eliminating $u_t, v_t, \phi_x, \phi_t, \psi_x, \psi_t, f_x, f_t$ in terms of the closed system, one arrive at a system of determining equations for the functions $X, T, U, V, P, Q,$ and F, which can be solved by using Maple to give

$$\begin{aligned}
X(x,t,u,v,\phi,\psi,f) &= c_4,\\
T(x,t,u,v,\phi,\psi,f) &= c_3,\\
U(x,t,u,v,\phi,\psi,f) &= c_2\phi^2 + c_1 u,\\
V(x,t,u,v,\phi,\psi,f) &= c_2\psi^2 - c_1 v,\\
P(x,t,u,v,\phi,\psi,f) &= \tfrac{(2c_2 f + c_1 + c_5)\phi}{2},\\
Q(x,t,u,v,\phi,\psi,f) &= \tfrac{(-2c_2 f + c_1 - c_5)\psi}{2},\\
F(x,t,u,v,\phi,\psi,f) &= c_2 f^2 + c_5 f + c_6,
\end{aligned} \tag{21}$$

where $c_i, \quad i = 1, 2, ..., 6$ are arbitrary constants.

4. Optimal system of the prolonged system

In general, to each s-parameter subgroup of the full symmetry group, there will correspond a family of group-invariant solutions. Because there are always an infinite number of subgroups, there is no need to list possible group-invariant solutions to the system. In this section, an optimal system of one-dimensional subalgebras of Eq.(2) by using the method presented in Refs.[2, 7] will be constructed.

As it is said in Ref.[2], the problem of finding an optimal system of subgroups is equivalent to finding an optimal system of subalgebras. From Eqs.(21), the associated vector fields for the one-parameter Lie group of infinitesimal transformations are six generators given by

$$\begin{aligned}
&v_1 = \tfrac{1}{2}\phi\tfrac{\partial}{\partial\phi} + \tfrac{1}{2}\psi\tfrac{\partial}{\partial\psi} + f\tfrac{\partial}{\partial f}, v_2 = \varphi^2\tfrac{\partial}{\partial u} + \psi^2\tfrac{\partial}{\partial v} + \phi f\tfrac{\partial}{\partial\psi} + f^2\tfrac{\partial}{\partial f},\\
&v_3 = \tfrac{\partial}{\partial f}, v_4 = u\tfrac{\partial}{\partial u} - v\tfrac{\partial}{\partial v} + \tfrac{1}{2}\phi\tfrac{\partial}{\partial\phi} - \tfrac{1}{2}\psi\tfrac{\partial}{\partial\psi}, v_5 = \tfrac{\partial}{\partial t}, v_6 = \tfrac{\partial}{\partial x},
\end{aligned} \tag{22}$$

One can know that v_4, v_5, v_6 are the centers of the group through calculating, so we don't have to consider them. Following Ref.[2], two subalgebras v_2 and v_1 of a given Lie algebra are equivalent if one can find an element g in the Lie group so that $Adg(v_1) = v_2$ where Adg is the adjoint representation of g on v. Given a nonzero vector, for example,

$$V = a_1 v_1 + a_2 v_2 + a_3 v_3,$$

where $a_j, \quad j = 1, 2, 3$ are arbitrary constants. The key task is to simplify as many of the coefficients a_i as possible though judicious applications of adjoint maps to v. In this way, one can get the following results in Table 1 where α is an arbitrary constant.

Table 1: Optimal Systems

Cases		Optimal systems
(a1)	$a_1 \neq 0,$	v_1
(a2)	$a_3 \neq 0,$	v_3
(a3)	$a_2 \neq 0, a_3 \neq 0, ,$	$v_2 + \alpha v_3$

5. Nonlocal conservation law of C-H equations

In this section, we briefly present the notations and theorems used in this paper firstly. Consider a system $F\{x; u\}$ of N partial differential equations of order s with n independent variables $x = (x^1, ..., x^n)$ and m dependent variables $u(x) = (u^1(x), ..., u^m(x))$, given by

$$F_\alpha[u] = F_\alpha(x, u, u_{(1)}, ..., u_{(s)}) = 0, \alpha = 1, ..., N \tag{23}$$

where $u_{(1)}$,...$u_{(s)}$ denote the collection of all first, . . ., sth-order partial derivatives. $u_i = D_i(u), D_{(ij)} = D_j D_i(u), ...$ Here $D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + ..., \quad i = 1, 2, ..., n.$

Definition 1: A conservation law of PDE system (23) is a divergence expression

$$D_i \Phi^i[u] = D_1 \Phi^1[u] + ... + D_n \Phi^n[u] = 0 \tag{24}$$

holding for all solutions of PDE system (23).

It is easy to see that Eqs.(14) determine a nonlocal conservation law, i.e.

$$D_t(f_x) + D_x(-f_t) = 0. \tag{25}$$

Definition 2:[22]. The adjoint equations of Eq. (1) is defined by

$$F^*_\alpha(x, u, v, u_{(1)}, m_{(1)}, ..., u_{(s)}, m_{(s)}) = E_{u^\alpha}(m^\beta F_\beta) = 0, \ \alpha = 1, 2, ..., N, \tag{26}$$

where $E_{u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{\mu=1}^{s}(-)^\mu D_{i_1}, ..., D_{i_\mu}\frac{\partial}{\partial u^\alpha_{i_1,...,i_\mu}}$ denotes the Euler-Lagrange operator, $m = m(x)$ is a new dependent variable $m = (m^1, m^2, ..., m^N)$.

Theorem 1 :[22]. The system consisting of Eqs.(23) and their adjoint Eqs.(26)

$$\begin{cases} F_\alpha(x, u, u_{(1)}, ..., u_{(s)}) = 0, \\ F^*_\alpha(x, u, m, u_{(1)}, m_{(1)}, ..., u_{(s)}, m_{(s)}) = 0, \end{cases} \tag{27}$$

has a formal Lagrangian

$$L = m^\beta F_\beta(x, u, u_{(1)}, ..., u_{(s)}). \tag{28}$$

Theorem 2:[22] Any Lie point, Lie-Bäcklund and non-local symmetry

$$V = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \tag{29}$$

of Eqs.(23)provides a conservation law $D_i(T^i) = 0$ for the system comprising Eqs.(23) and its adjoint Eqs.(26). The conserved vector is given by

$$T^i = \xi^i L + W^\alpha E_{u^\alpha_i}(L) + D_j(W^\alpha)E_{u^\alpha_{ij}}(L) + D_jD_k(W^\alpha)E_{u^\alpha_{ijk}}(L) + ..., \tag{30}$$

where W^α is the Lie characteristic function

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j,$$

and L is determined by Eqs.(28).

Next, we construct the conservation laws Eqs(3). Let us consider the prolonged system, one can obtain the following results from the above,

$$\begin{aligned} &\xi^1 = c_3, \xi^2 = c_4, \eta^1 = c_2\varphi^2 + c_1 u, \eta^2 = c_2\phi^2 - c_1 v, \\ &\eta^3 = \tfrac{(2c_2 f + c_1 + c_5)\varphi}{2}, \eta^4 = \tfrac{(-2c_2 f + c_1 - c_5)\phi}{2}, \eta^5 = c_2 f^2 + c_5 f + c_6. \end{aligned}$$

and

$$\begin{aligned} L = {} & m^1(iu_t + \alpha(u_{xx} - 2u^2 v) + i\beta(u_{xxx} - 6uvu_x)) + m^2(iv_t - \alpha(v_{xx} - 2v^2 u) + i\beta(v_{xxx} - 6uvv_x)) + m^3(-i\lambda\phi + u\psi) \\ & + m^4(v\phi + i\lambda\psi) + m^5(a\phi + b\psi) + m^6(c\phi - a\psi) + m^7(\varphi\phi) + m^8(-\beta\varphi^2 v_x + 12\beta\lambda^2\varphi\phi + 4\lambda\alpha\varphi\phi - \beta\phi^2 u_x + 2\beta\varphi\phi uv \\ & + 4i\lambda\beta\phi^2 u - 4i\lambda\beta\varphi^2 v + i\alpha\phi^2 u - i\alpha\varphi^2 v) \end{aligned}$$

and a, b, c are determined by Eqs.(5).

Using the theorem 2, one can get the following results by calculation,

$$
\begin{aligned}
T^1 &= m^1(c_3\beta u_{xxx} + 2i\alpha c_3 u^2 v + c_2\phi^2 - c_4 u_x + 2c_1 u - c_3 i\alpha u_{xx} - 6c_3\beta uvu_x) + m^2(-6c_3\beta uvv_x + c_2\psi^2 - c_4 v_x \\
&-2c_1 v + c_3\beta v_{xxx} - 2ic_3\alpha uv^2 + c_3\alpha iv_{xx}) + m^4(c_3 i\alpha\phi uv + c_3\beta u\phi v_x - 2c_3\beta u^2 v\psi - 4c_3\beta\lambda^2\psi u - c_4\psi u - 2c_3\alpha\lambda u\psi \\
&+c_3\beta\psi u_{xx} + c_4 i\lambda\phi - c_3 i\alpha\psi u_x - c_3\beta\phi vu_x + 2ic_3\beta\lambda uv\phi + 2ic_3\alpha\phi\lambda^2 c_2 f\phi + 4ic_3\beta\phi\lambda^3 - 2ic_3\beta\lambda\psi u_x + c_5\phi + c_1\phi) \\
&+m^6(-c_4 i\lambda\psi - 2ic_3\beta\lambda\psi uv + c_3\beta v\psi u_x - 4c_3\beta\phi v\lambda^2 + c_5\psi + c_2 f\psi + 2ic_3\beta\lambda\phi v_x - c_3 i\alpha uv\psi - c_4 v\phi + c_3\beta\phi v_{xx} \\
&-2c_3\alpha\lambda\phi v - 2c_3\beta\phi uv^2 - 4ic_3\beta\lambda^3\psi + c_3 i\alpha\phi v_x - c_3\beta u\psi v_x - 2c_3 i\alpha\lambda^2\psi - c_1\psi) + m^8(-2c_3\beta\phi uv\psi - 4ic_3\beta\lambda u\psi^2 \\
&-c_4\phi\psi - 12c_3\beta\lambda^2\phi\psi - 4c_3\alpha\lambda\phi\psi - c_3 i\alpha u\psi^2 + c_2 f^2 + 2c_5 f + c_6 + c_3\beta\phi^2 v_x + c_3\beta\psi^2 u_x + 4ic_3\beta\lambda v\phi^2 + c_3 i\alpha v\phi^2)
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
T^2 &= m^7(c_2 f^2 + 2c_5 f + c_6 - c_3 f_t - c_4\psi\phi) + m^5(-c_3\psi_t + c_5\psi - c_1\psi - c_4 v\phi - c_4 i\lambda\psi + c_2 f\psi) + m^8(2c_2\beta\psi^2\phi^2 \\
&c_3\beta\phi^2 v_t + 2c_1\beta u\psi^2 - 2c_1\beta v\phi^2 + c_4 f_t - 2c_4\beta\psi\phi uv - 4ic_4\beta\lambda u\psi^2 + 4ic_4\beta\lambda v\phi^2 + c_4 i\alpha v\phi^2 - 12c_4\beta\lambda^2\psi\phi \\
&-4c_4\lambda\alpha\psi\phi - c_4 i\alpha u\psi^2 - c_3\beta\psi^2 u_t) + m^6(c_4\psi_t - 4ic_1\beta\lambda v\phi - c_4 i\alpha uv\psi - 2ic_3\beta\lambda\phi v_t + 2ic_2\beta\lambda\psi^2\phi + c_2\beta v\psi\phi^2 \\
&+4c_1\beta uv\psi + c_2 i\alpha\psi^2\phi - c_3\beta v\psi u_t + c_3\beta u\psi v_t + 2c_2\beta\psi\phi\psi_x - 2c_4\beta uv^2\phi - 4c_4\beta\lambda^2 v\phi - 2c_4\alpha\lambda v\phi - 2ic_1\alpha v\phi \\
&-i\alpha c_3\phi v_t - 4ic_4\beta\lambda^3\psi - 2ic_4\lambda^2\alpha\psi - 2ic_4\lambda\beta uv\psi - c_2\beta\psi^2\phi_x + c_3\beta v_t\phi_x + c_4\beta v_x\phi_x - 2c_1\beta v_x\phi - c_2\beta u\psi^3 \\
&-c_3\beta\phi v_{xt}) + m^4(c_4\phi_t - 4c_1\beta uv\phi + c_2\beta u\psi^2\phi - c_3\beta u\phi v_t + 2c_2\beta\phi\psi\phi_x - 2c_4\beta u^2 v\psi - 4c_4\beta\lambda^2 u\psi - 2c_4\lambda\alpha u\psi \\
&+c_3\beta v\phi u_t + c_3 i\alpha\psi u_t - c_2 i\alpha\psi\phi^2 - 2c_1 i\alpha u\psi + 4ic_4\beta\lambda^3\phi + 2ic_4\alpha\lambda^2\phi + ic_4\alpha uv\phi + 2ic_3\beta\lambda\psi u_t - 2ic_2\beta\lambda\psi\phi^2 \\
&-4ic_1\beta\lambda\psi u + 2ic_4\beta\lambda uv\phi - c_3\beta\psi u_{xt} - c_2\beta\phi^2\psi_x - 2c_1\beta u\psi_x + c_3\beta\psi_x u_t + c_4\beta\psi_x u_x + 2c_1\beta\psi u_x - c_2\beta\phi^3 v) \\
&+m^2(2c_2\beta\psi\psi_{xx} - 6c_2\beta uv\psi^2 - 2c_1\beta v_{xx} + 12c_1\beta uv^2 - ic_3\alpha v_{xt} + 2ic_2\alpha\psi\psi_x - 2ic_4\alpha uv^2 + c_4 v_t + 6c_3\beta uvv_t \\
&-2ic_1\alpha v_x + 2c_2\beta\psi_x^2 - c_3\beta v_{xxt}) + m^1(-2ic_1\alpha u_x + c_4 u_t + ic_3\alpha u_{xt} + 2c_1\beta u_{xx} + 2c_2\beta\phi\phi_{xx} + 6c_3\beta uvu_t - 12c_1\beta vu^2 \\
&-2ic_2\alpha\phi\phi_x + 2c_2\beta\phi_x^2 - c_3\beta u_{xxt} - 6c_2\beta uv\phi^2 + 2ic_4\alpha u^2 v) + m^3(-c_4 u\psi + c_5\phi + ic_4\lambda\phi + c_1\phi + c_2\phi f - c_3\phi_t) - \\
&2c_1\beta u_x m_x^1 + 2c_1\beta v_x m_x^2 + ic_2\alpha\phi^2 m_x^1 - c_2\beta\psi\phi^2 m_x^4 - 2c_1\beta u\psi m_x^4 - ic_4\alpha u_x m_x^1 - ic_2\alpha\psi^2 m_x^2 + 2i\alpha c_1 v m_x^2 + \\
&2i\alpha c_1 u m_x^1 + c_4\beta\psi u_x m_x^4 + ic_4\alpha v_x m_x^2 + ic_3\alpha v_t m_x^2 - c_2\beta\psi^2\phi v_x m_x^6 + c_4\beta\phi v_x m_x^6 + c_3\beta\phi v_t m_x^6 + 2c_1\beta\phi v m_x^6 \\
&-2c_2\beta\phi\phi_x m_x^1 - 2c_2\beta\psi\psi_x m_x^2 - c_3\beta v_t m_{xx}^2 + c_3\beta v_{xt} m_x^2 - c_3\beta u_t m_{xx}^1 + c_3\beta u_{xt} m_x^1 + c_2\beta\phi^2 m_{xx}^1 + 2c_1\beta u m_{xx}^1 \\
&-c_4\beta u_x m_{xx}^1 + c_2\beta\psi^2 m_{xx}^2 - c_4\beta v_x m_{xx}^2 - 2c_1\beta v m_{xx}^2 + c_4\beta m_x^1 u_{xx} + c_4\beta m_x^2 v_{xx} - ic_3\alpha m_x^1 u_t + c_3\beta\psi m_x^4 u_t
\end{aligned}
\tag{32}
$$

where $m_t^1, m_t^2, m_x^3, m_x^5, m_x^7$ can be obtained by using (26) and (28). Because the formulas are complicated, so here we omit.

Through the verification, T^1, T^2 satisfy the equation(24),i.e.

$$
D_t(T^1) + D_x(T^2) = 0,
\tag{33}
$$

thus, Eqs.(31),(32) define the corresponding components of a non-local conservation law for the system of Eqs. (2).

Remark1: For the coupled Hirota equations (2), we get its conservation laws by making use of the explicit solution of the adjoint equations of Eqs. (2). But the number of the adjoint equations is less than the number of variables, so T^1, T^2 have arbitrary functions, we can conclude that Eqs.(2) have infinitely many conservation laws.

6. Summary and Discussion

In this paper, the nonlocal symmetry of coupled Hirota equations is obtained by using the lax pair and localized by introducing an auxiliary dependent variable. Then, the primary nonlocal symmetry is equivalent to a Lie point symmetry of a prolonged system. On the basis of this system, the one-dimensional subalgebras of a Lie algebra have been classified and the reductions of coupled Hirota equations are given out by using the associated vector fields. On the one hand, non-trivial nonlocal conservation laws of the coupled Hirota equation are obtained by means of these formulas. For general evolution equation, the nonlocal conservation laws may be not enough to guarantee the integrability. However, it must be useful to help to understand the special solutions of the nonlinear systems.

This method would be possible to extend to many other interesting integrable models. However, there is not a universal way to estimate what kind of nonlocal symmetries can be localized to some related prolonged system, so there are still many questions worth to study. Moreover, one can construct infinitely many nonlocal symmetries by introducing some internal parameters from the seed symmetry and infinitely many nonlocal conservation laws of the completely integrable finite-dimensional systems. Above topics will be discussed in the future series research works.

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